Lecture 7
Wake fields and beam instabilities. Beam break-up instability

January 24, 2019

Lecture outline

- Accelerator models and wake fields in the equations of motion
- Beam break-up instability
- Effect of acceleration
- BNS damping, authophasing

The accelerator model

We will now apply the notion of wake fields to the problem of beam instabilities.

Recall the equations of motion of particles in accelerators. Here we will consider the transverse motion only — the longitudinal motion is studied in L9.

We will use a simplified version of the linear betatron oscillations,

$$\frac{d^2x}{dt^2} + \omega_x^2 x = 0, \qquad \frac{d^2y}{dt^2} + \omega_y^2 y = 0 \tag{7.1}$$

In this model $\omega_x/c = 1/\beta_x$, $\omega_y/c = 1/\beta_y$. We will also assume $\omega_x = \omega_y = \omega_\beta$. The time derivative is often replaced by the derivative along the orbit coordinate s,

$$\frac{d}{dt} = \beta c \frac{d}{ds} \approx c \frac{d}{ds}$$

How do we add wakes to these equations? See Eqs. (3.1).

Adding transverse wakes

With the transverse wake *per unit length* w_t, the quantity $q^2 w_t$ is the change of the momentum per unit time, so

$$\frac{d^2 x}{dt^2} + \omega_x^2 x = \frac{q^2}{m\gamma} W_x, \qquad \frac{d^2 y}{dt^2} + \omega_y^2 y = \frac{q^2}{m\gamma} W_y \tag{7.2}$$

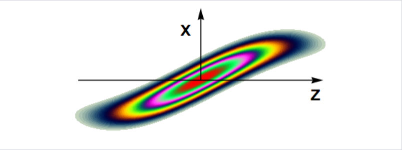
Here W_x and W_y are the wakes of all the charges in the beam at the location of a given particle.
In general, wakes depend on the coordinates of the test particles, as well as coordinates of the source particles and the time. Usually we neglect the dependence of the transverse wake on the coordinates of the test particle (which is true for the axisymmetric systems), and only take into account its dependence on the coordinate of the source particles.
Assuming that we know the longitudinal particle density in the bunch, $\lambda(z)$, normalized by unity, $\int_{-\infty}^{\infty} \lambda(z) dz = 1$,

$$W(z) \to N \int_z^{\infty} w(z' - z) \lambda(z') dz' \tag{7.3}$$

where N is the number of particles in the bunch (remember, z is measured in the direction of motion of the beam).

Beam Break-up Instability

If the beam is injected with an offset in a linac, the head of the bunch executes betatron oscillations and generates the transverse wake that drives the tail of the bunch. If the betatron frequency in the tail is the same as in the head, this is a resonant effect. As a result, the amplitude of the betatron oscillations in the tail grows with distance along the linac. The is called *the beam breakup instability* — BBU. Often BBU is introduced via a two-particle model (see A. Chao's book).



We now assume that the beam energy remains constant (no acceleration). All particles oscillate with the same betatron frequency. See movie 6.

The equation for the betatron oscillaitons ($q \to e$)

$$\frac{\partial^2 x(t,z)}{\partial t^2} + \omega_\beta^2 x(t,z) = \frac{Ne^2}{\gamma m} \int_z^\infty dz' x(t,z') \bar{w}_t(z'-z) \lambda(z') \qquad (7.4)$$

We will assume that initially there is a uniform offset of the beam, $x(0,z) = x_0$.

Slow-amplitude approximation

The wake field is typically weak — its effect is much smaller than the betatron frequency,

$$\frac{Ne^2}{\gamma m}\bar{w}_t \ll \omega_\beta^2$$

Seek solution in the complex form

$$x(t,z) = a(t,z)e^{-i\omega_\beta t}$$

where a is a "slow" function that varies on a time scale T much larger than ω_β^{-1}, $\omega_\beta T \gg 1$. We neglect $\partial^2 a/\partial t^2 \sim a/T^2$ in comparison with $\omega_\beta \partial a/\partial t \sim \omega_\beta a/T$,

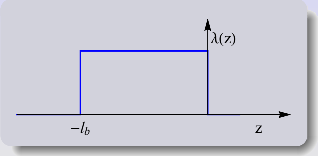
$$\frac{\partial^2 x}{\partial t^2} \approx -2i\omega_\beta \frac{\partial a}{\partial t} e^{i\omega_\beta t} - \omega_\beta^2 x$$

The equation for the amplitude a is

$$\frac{\partial a}{\partial t} = i\frac{Ne^2}{2\omega_\beta m\gamma}\int_z^\infty dz' a(t,z')\bar{w}_t(z'-z)\lambda(z') \qquad (7.5)$$

Case of constant wake and boxcar distribution

There are two cases where the BBU equation is solvable analytically. First, assume that the beam linear density is constant, $\lambda(z) = 1/\ell_b$ and that the wake $\bar{w}_t$ is constant (does not depend on z). Differentiate (7.5) over z:



$$\frac{\partial^2 a}{\partial t \partial z} = -\frac{1}{2} ira$$

where

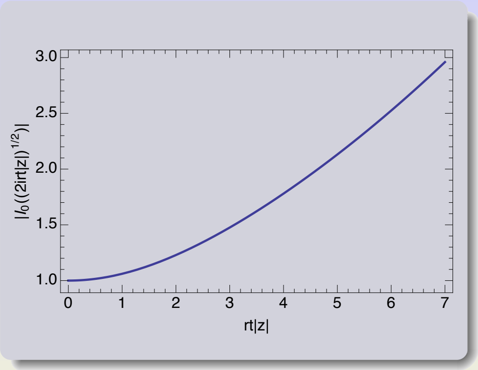
$$r = \frac{Ne^2 \bar{w}_t}{\omega_\beta \gamma \ell_b m}$$

Solution of this equation with an initial condition $a = x_0$ at $t = 0$ is

$$a = x_0 I_0 \left(\sqrt{-2irtz} \right) = x_0 I_0 \left(\sqrt{2irt|z|} \right)$$

where I_0 is the modified Bessel function of the zeroth order (*see the proof in the Mathematica notebook*).

Amplitude grows with time



Asymptotically, for $rt|z| \gg 1$, the amplitude grows exponentially

$$|a| \sim \frac{x_0}{2^{3/4}\pi^{1/2}} \frac{1}{(rt|z|)^{1/4}} \exp\left(\sqrt{rt|z|}\right)$$

This dependence differs from the circular accelerator instabilities where typically instability (Γ is the inverse growth time) $\propto e^{\Gamma t}$.

Case of a linear wake

In the second case, we assume a linear wake, $\bar{w}_t = \bar{w}_t' z$. Recall that the linear transverse wake, Eq. (5.18), is universally correct for short distances. This case was first studied by Chao, Richter and Yao in 1980. Again assume a rectangular beam profile shown two slides ago. We now define the parameter r as

$$r = \frac{Ne^2 \bar{w}_t'}{2\omega_\beta \gamma \ell_b m}$$

The equation for the amplitude a of the oscillations is

$$\frac{\partial a}{\partial t} = -ir \int_z^0 dz'(z' - z) a(t, z')$$

This equation does not have an analytical solution, but one can find that asymptotically, for $t \to \infty$, the amplitude a grows as

$$|a| \sim \frac{x_0}{\sqrt{6\pi}} (2rtz^2)^{-1/6} \exp\left[\frac{3^{3/2}}{4} (2rtz^2)^{1/3}\right]$$

Computer simulations of BBU

Simulation of a Gaussian bunch. We start with Eq. (7.5) and absorb all dimensional factors into normalization of t, $\tilde{t}$ is the normalized time,

$$\frac{\partial a}{\partial \tilde{t}} = i \int_{z}^{\infty} dz' a(\tilde{t}, z') \tilde{w}_t(z' - z) \tilde{\lambda}(z') \tag{7.6}$$

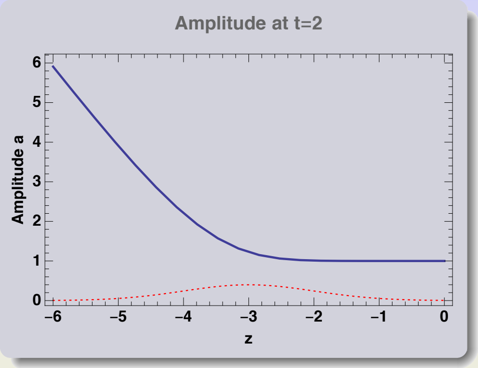
Split the bunch in slices in z: $z_1, z_2, \ldots, z_n$ with z_1 at the head of the bunch (in the code $z_1 = 0$ and the subsequent z_n are negative). Integrate the equation over t, sum over preceding slices. The amplitude of the first slice does not change, which means that $a(\tilde{t}, z_1) = \mathrm{const}$.

Computer simulations of BBU

```
BBU[t_, nSlice_, windowZ_] := Module[{Δz, z, aInteg, a},
  Δz = windowZ/(nSlice - 1);
  z = Table[-(windowZ * i)/(nSlice - 1), {i, 0, nSlice - 1}];
  a[ξ_, 1] = 1.0;
  Do[aInteg[ξ_, i - 1] = Integrate[a[t', i - 1], {t', 0, ξ}];
   a[ξ_, i] = 1 + Sum[ⅈ Δz aInteg[ξ, k] ρ[z[[k]]] w[z[[k]] - z[[i]]], {k, 1, i - 1}] ,
   {i, 2, nSlice}];
  Return[Table[{z[[i]], a[t, i]}, {i, 1, nSlice}]];
 ]
w[z_] := z;
ρ[z_] := 1/Sqrt[2 π] Exp[-1/2 (z + 3)^2]; (*σ=1*)
bbu = BBU[t, 20, 6];
```

(an example of quick but bad programming style!)

Computer simulations of BBU



Linear acceleration of the beam

We neglected the fact that the beam is being accelerated in the linac. We now take this into account:

$$\gamma(t) = \gamma_i + ct\frac{\gamma_f - \gamma_i}{L}$$

Here L is the linac length, $\gamma_i mc^2$ is the initial energy, $\gamma_f mc^2$ is the final energy. The equation of motion needs to be modified

$$\frac{dp_x}{dt} = mc\frac{d}{dt}\gamma v_x \approx mc^2\frac{d}{dt}\gamma\frac{dx}{dt}$$

$$\frac{1}{\gamma}\frac{\partial}{\partial t}\gamma\frac{\partial x(t,z)}{\partial t} + \omega_\beta^2 x(t,z) = \frac{Ne^2}{\gamma m}\int_z^\infty dz' x(t,z')\bar{w}_t(z'-z)\lambda(z')$$

where $\gamma = \gamma(t)$. We will assume a "slow" acceleration, $\dot{\gamma}/\gamma \ll \omega_\beta$.

Linear acceleration of the beam

First let's see what we have without the wake

$$\frac{\partial^2 x}{\partial t^2} + \frac{\dot{\gamma}}{\gamma}\frac{\partial x}{\partial t} + \omega_\beta^2 x = 0$$

Again, seek solution in the form $x(t, z) = a(t, z)e^{-i\omega_\beta t}$ and neglect the second derivative $\partial^2 a/\partial t^2$ as well as the term $\propto \dot{\gamma}\partial a/\partial t$,

$$-2i\omega_\beta \frac{\partial a}{\partial t} - \frac{\dot{\gamma}}{\gamma} i\omega_\beta a = 0$$

The solution

$$a(t, z) = a_0(0, z)\sqrt{\frac{\gamma_0}{\gamma(t)}}$$

We found the *adiabatic damping* of the betatron oscillations.

Linear acceleration of the beam

We now take the wake into account,

$$-2i\omega_\beta \frac{\partial a}{\partial t} - \frac{\dot{\gamma}}{\gamma} i\omega_\beta a = \frac{Ne^2}{\gamma m} \int_z^\infty dz' a(t,z') \bar{w}_t(z'-z)\lambda(z')$$

Introduce a new variable b, $a(t,z) = b(t,z)/\sqrt{\gamma(t)}$.

$$-2i\omega_\beta \frac{\partial b}{\partial t} = \frac{Ne^2}{\gamma(t) m} \int_z^\infty dz' b(t,z') \bar{w}_t(z-z')\lambda(z')$$

This is the same equation as (7.5), except for $\gamma(t)$. This extra $\gamma(t)$ can be eliminated if we define the new "time" τ,

$$d\tau = dt \frac{1}{\gamma(t)}$$

Linear acceleration of the beam

$$-2i\omega_\beta \frac{\partial b}{\partial \tau} = \frac{Ne^2}{m} \int_z^\infty dz' b(\tau, z') \bar{w}_t(z' - z) \lambda(z')$$

This is our old equation without the acceleration.
For linear acceleration, the equation for τ can be easily integrated. The value of τ that corresponds to the end of the linac, τ_L, is

$$\tau_L = \frac{L/c}{\gamma_f - \gamma_i} \ln \frac{\gamma_f}{\gamma_i} \approx \frac{L}{c\gamma_f} \ln \frac{\gamma_f}{\gamma_i}$$

If we know a solution for $\gamma = \mathrm{const}$, then for an accelerating beam

$$\frac{L}{\gamma} \to \frac{L}{\gamma_{\mathrm{eff}}}$$

where $\gamma_{\mathrm{eff}} = \gamma_f / \ln \frac{\gamma_f}{\gamma_i}$.

BBU instability

With account of the acceleration, for the linear wake, we now (asymptotically) have

$$\frac{|a|}{x_0} \sim \frac{1}{\sqrt{6\pi}}(2rsz^2)^{-1/6}\exp\left[\frac{3^{3/2}}{4}(2rsz^2)^{1/3}\right]$$
$$\to \frac{1}{\sqrt{6\pi}}\left(\frac{\Upsilon}{3^{3/2}/4}\right)^{-1/2}\left(\frac{\gamma_i}{\gamma_f}\right)^{1/2} e^{\Upsilon}$$

where the parameter "upsilon"

$$\Upsilon = \frac{3^{3/2}}{4}\left(\frac{4\pi}{Z_0 c}\frac{N r_e \bar{w}_t' L \ell_b}{k_\beta \gamma_{\mathrm{eff}}}\right)^{1/3}$$

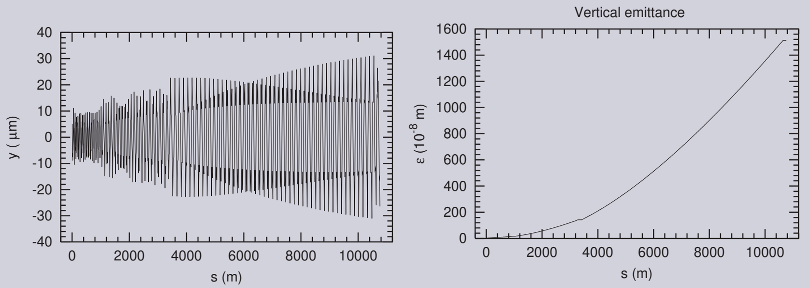
(we included the adiabatic damping, $a_0 \to a_0\sqrt{\gamma_i/\gamma_f}$.

BBU simulations for NLC with LIAR

BBU in a linear collider (NLC).

$N_{\rm part} = 1.1 \times 10^{10}$
$L = 10$ km
$\ell_b = 1.5 \times 150$ micron
$E_i = 10$ GeV, $E_f = 250$ GeV
$\omega_\beta / c = 1/30$ m^{-1}
$\bar{w}_t' = 100 \ \frac{\rm V}{\rm pC \cdot m \cdot mm^2}$

LIAR is a computer code developed at SLAC for simulation of beam dynamics in linear colliders. Initial beam offset 5 micron, no energy spread, $\epsilon_{y0} = 3 \cdot 10^{-8}$ m.



BNS damping

Balakin, Novokhatski and Smirnov (BNS) in 1983 proposed how to suppress the BBU — the *BNS damping*.

The idea: break the resonant interaction in the BBU by introducing a variation of the betatron frequency in the bunch. This can be achieved by varying the energy of particles in the bunch from head to tail.

In the FODO lattice the phase advance μ of the betatron oscillations (per cell) depends on the energy deviation ($\eta = \Delta E/E$)

$$\frac{d\mu}{d\eta} = -2\tan\frac{\mu}{2}$$

In a smooth focusing approximation, $\mu/\ell_{\rm cell} \to \omega_\beta/c$, hence $\omega_\beta(\eta)$ with

$$\frac{1}{c}\frac{d\omega_\beta}{d\eta} \to -\frac{2}{\ell_{\rm cell}}\tan\frac{\mu}{2}$$

We can introduce a linear *energy chirp* in the beam by accelerating it off-crest:

$$\eta(z) = Az\,, \qquad \omega_\beta = \omega_{\beta,0} + \Delta\omega_\beta(z)$$

BNS damping

The betatron oscillations with account of the energy chirp and the wake,

$$\frac{\partial^2 x(t,z)}{\partial t^2} + [\omega_{\beta,0} + \Delta\omega_\beta(z)]^2 x(t,z) = \frac{Ne^2}{\gamma m} \int_z^\infty dz' x(t,z') \bar{w}_t(z'-z) \lambda(z')$$

Assume again that

$$\Delta\omega_\beta, \;\; \frac{Ne^2}{\gamma m \omega_{\beta,0}} \bar{w}_t \ll \omega_{\beta,0}$$

Neglect $\Delta\omega_\beta^2$

$$\frac{\partial^2 x(t,z)}{\partial t^2} + [\omega_{\beta,0}^2 + 2\omega_{\beta,0}\Delta\omega_\beta(z)] x(t,z) = \frac{Ne^2}{\gamma m} \int_z^\infty dz' x(t,z') \bar{w}_t(z'-z) \lambda(z')$$

BNS damping

Assume that the beam is offset initially

$$x(0, z) = x_0$$

Can we find such $\Delta\omega_\beta(z)$, that x does not depend on z at a later time, $x(t, z) = X(t)$?

$$2\omega_{\beta,0}\Delta\omega_\beta(z) = \frac{Ne^2}{\gamma m}\int_z^\infty dz' \bar{w}_t(z' - z)\lambda(z')$$

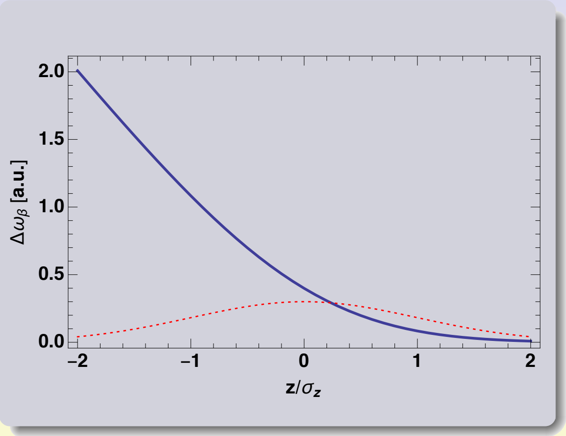
$$\frac{\partial^2 X(t)}{\partial t^2} + \omega_{\beta,0}^2 X(t) = 0$$

This is called the *autophasing*:

$$\Delta\omega_\beta \sim \frac{Ne^2 \bar{w}_t}{\gamma m \omega_{\beta,0}}$$

BNS damping

The authophasing requires a special profile of $\Delta\omega_\beta(z)$ that depends on the wake and the distribution function of the beam. Example: the Gaussian beam distribution with a linear wake.



More focusing at the tail, less at the head

BNS damping

Even without the autophasing the energy chirp in the beam can help to suppress the instability

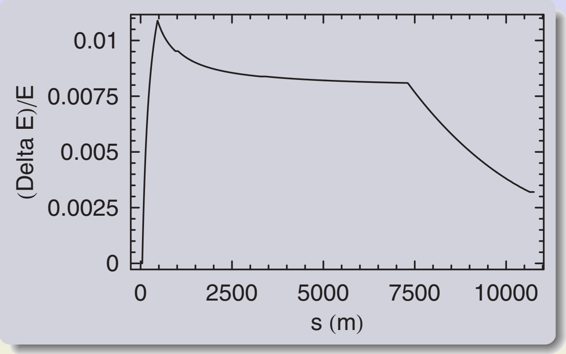
$$x(t,z) = a(t,z)e^{-i[\omega_{\beta,0}+\Delta\omega_\beta(z)]t}$$
$$\frac{\partial^2 x(t,z)}{\partial t^2} \approx -2i\omega_{\beta,0}\frac{\partial a}{\partial t} - [\omega_{\beta,0}+\Delta\omega_\beta(z)]^2 a$$

$$\frac{\partial a}{\partial t} = i\frac{Ne^2}{2\omega_{\beta,0}m\gamma}\int_z^\infty dz' a(t,z')\bar{w}_t(z'-z)\lambda(z')e^{-it\Delta\omega_\beta(z)}$$

movie7 shows the instability with and without BNSCou

LIAR simulation of NLC beam dynamics

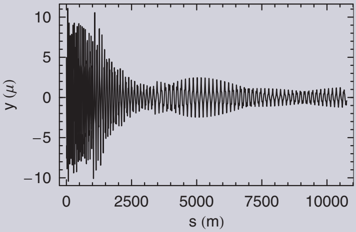
Energy chirp in the beam for the BNS damping

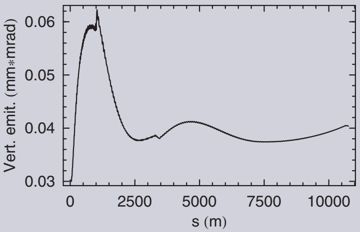


The energy spread is generated by a proper choice of RF phases:

ϕ_1 [deg.]	E_1 [GeV]	ϕ_2 [deg.]	E_2 [GeV]	ϕ_3 [deg.]
12	30	-1	355	-30

LIAR simulation of NLC beam dynamics





With the BNS energy chirp an initial offset in the beam damps down along the linac, and the associated emittance growth is limited.

The price for the BNS damping:

- lower acceleration gradient, $\sim 3\%$
- severe tolerances on quadrupole misalignment